Lett Math Phys (2009) 89:85–100
DOI 10.1007/s11005-009-0326-3

On the Space of KdV Fields

ATSUSHI NAKAYASHIKI
Department of Mathematics, Kyushu University, Ropponmatsu 4-2-1, Fukuoka 810-8560, Japan. e-mail: 6vertex@math.kyushu-u.ac.jp

Received: 3 April 2009 / Revised: 11 May 2009 / Accepted: 27 May 2009
Published online: 16 June 2009 –

Abstract. The space of functions A over the phase space of KdV-hierarchy is studied as a module over the ring $\mathcal{D}$ generated by commuting derivations. A $\mathcal{D}$-free resolution of A is constructed by Babelon, Bernard and Smirnov by taking the classical limit of the construction in quantum integrable models assuming a certain conjecture. We propose another $\mathcal{D}$-free resolution of A by extending the construction in the classical finite dimensional integrable system associated with a certain family of hyperelliptic curves to infinite dimension assuming a similar conjecture. The relation between the two constructions is given.

Mathematics Subject Classification (2000). 37K20, 17B69, 17B37.

Keywords. KdV hierarchy, boson-fermion correspondence, space of fields, D-module.

1. Introduction

In [1] Babelon, Bernard and Smirnov (BBS), by considering the classical limit of a model in two dimensional integrable quantum field theory, have studied the space of KdV fields $A=\mathbb{C}[u,u',\ldots]$ as a module over the ring of commuting derivations $\mathcal{D}=\mathbb{C}[\partial_1,\partial_3,\ldots]$, where ∂_i acts on u according as the KdV-hierarchy (see Section 2):

$$\partial_i u = S'_{i+1}(u), \quad S_{i+1}(u)\in A.$$

Assuming the conjecture that A is generated over $\mathcal{D}$ by $\mathbb{C}[S_2,S_4,\ldots]$ they have constructed a $\mathcal{D}$-free resolution of A. In particular all $\mathcal{D}$-linear relations among monomials of $\{S_{2i}\}$ are determined. They are called null vectors in [1]. For example the first two non-trivial null vectors are

$$\partial_3 S_2-\partial_1 S_4=0, \quad \partial_1^2 S_2-4S_4+6S_2^2=0, \tag{1}$$

which give the KdV equation for S_2:

$$\partial_3 S_2=\frac{1}{4}\partial_1^3 S_2+3S_2\partial_1(S_2).$$

In [11] the affine ring A_g of the affine Jacobian of a hyperelliptic curve of genus g is studied as a module over the ring $\mathcal{D}_g = \mathbb{C}[\partial_1, \partial_3, \ldots, \partial_{2g-1}]$ of invariant vector fields on the Jacobian. Assuming some conjecture a $\mathcal{D}_g$-free resolution of A_g has been constructed. Although the conjecture is verified only for $g \leq 3$ [9] up to now, this construction exhibits a remarkable consistency with other results. For example it recovers the character of A_g [11] and the cohomologies of the affine Jacobian [7]. Since the $g \to \infty$ limit of A_g is identified with A, in the present paper we directly construct a $\mathcal{D}$-free resolution of A extending the construction for A_g. Let $\tau(t)$ be the tau function of the KdV-hierarchy and $\zeta_{i_1 \ldots i_n} = \partial_{i_1} \cdots \partial_{i_n} \log \tau(t)$, $\partial_i = \partial/\partial t_i$. Then the generators of A over $\mathcal{D}$ of this construction are given by the set of functions

$$1, \quad (i_1 \ldots i_n; j_1 \ldots j_n) := \det(\zeta_{i_k, j_l})_{1 \leq k, l \leq n}, \quad n \geq 1,$$

where $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n$. The $\mathcal{D}$-linear relations among them are, for example,

$$\partial_{j_1}(i_1 i_2 : j_2 j_3) - \partial_{j_2}(i_1 i_2 : j_1 j_3) + \partial_{j_3}(i_1 i_2 : j_1 j_2) = 0.$$

In a sense these are trivial relations since they hold if $\tau(t)$ is replaced by an arbitrary function of $t_1, t_3, \ldots$ Notice that $S_{2n} = \partial_1 \partial_{2n-1} \log \tau(t)$ and the null vectors of BBS implies the bilinear form of the KdV equation for $\tau(t)$. Thus two free resolutions of A give quite different generators and relations. However, we prove that two constructions are equivalent by showing the equivalence of two conjectures.

The construction of [1] is related with the quantum groups at root of unity [6,8]. While the construction of the present paper is directly related with the geometry of Jacobian varieties and the analysis of abelian functions [9,10]. Thus the present result opens the way to study the latter subjects in terms of the representation theory. It is a quite interesting problem to extend conjectures and results of [9,11] to the case of more general algebraic curves than that of hyperelliptic curves based on this view point. Toward this direction the results of [4,12,13] are important.

The present paper is organized in the following manner. In Section 2 the space of KdV fields is defined. After reviewing the boson-fermion correspondence in Section 3, the construction of the free resolution of A due to Babelon et al. is reviewed in Section 4. In Section 5 another construction of the free resolution of A is given. The relation of two constructions is given in Section 6. Finally in Section 7 concluding remarks are given.

2. The Space of KdV Fields

Let A denote the differential algebra

$$A = \mathbb{C}[u, u', u'', \ldots] \tag{2}$$

generated by $u = u^{(0)}$, $u' = u^{(1)}$, $u'' = u^{(2)}$, ...such that the derivation$'$ acts as $(u^{(m)})' = u^{(m+1)}$ for any m. The KdV hierarchy is the infinite number of compatible differential equations given by

$$\frac{\partial u}{\partial t_n} = S'_{n+1}(u), \quad n = 1, 3, 5, \ldots, \tag{3}$$

where $S_n(u)$ is the element of A without constant term satisfying the equation

$$S'_{n+2}(u) = \frac{1}{4}S'''_n(u) - uS'_n(u) - \frac{1}{2}u'S_n, \quad S_2(u) = -\frac{1}{2}u. \tag{4}$$

In particular $'$ is identified with $\partial/\partial t_1$. The KdV hierarchy defines the action of the commuting derivations ∂_n on A by

$$\partial_n(u^{(k)}) = S^{(k+1)}_{n+1}(u), \tag{5}$$

Thus A is a $\mathcal{D}$-module, where $\mathcal{D} = \mathbb{C}[\partial_1, \partial_3, \cdots]$.

3. Free Fermions and Fock Spaces

Let ψ_n, ψ^*_n, $n \in 2\mathbb{Z}+1$, satisfy the relations

$$[\psi_m, \psi_n]_+ = [\psi^*_m, \psi^*_n]_+ = 0, \quad [\psi^*_m, \psi_n]_+ = \delta_{m,n}, \tag{6}$$

where $[X, Y]_+ = XY + YX$. The vacuums $<m|$, $|m>$, $m \in 2\mathbb{Z}+1$ are defined by the conditions

$$\begin{aligned} <m|\psi_n = 0 \quad \text{for } n \le m, \quad <m|\psi^*_n = 0 \quad \text{for } n > m,\\ \psi_n|m> = 0 \quad \text{for } n > m, \quad \psi^*_n|m> = 0 \quad \text{for } n \le m. \end{aligned}$$

They are related by

$$\psi^*_m|m-2> = |m>, \quad <m-2|\psi_m = <m|.$$

The Fock spaces H_m, H^*_m are constructed from $|m>$ and $<m|$ respectively by the equal number of ψ_k and ψ^*_l. The pairing between H_m and H^*_m are defined by normalizing

$$<m|m> = 1.$$

Let us set

$$h_{-2k} = \sum_{n \in 2\mathbb{Z}+1} \psi_n \psi^*_{n+2k},$$
$$T = \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} J_{2k} h_{-2k}\right),$$

where J_{2k} are commutative variables. Notice that $<m|T=<m|$ for any m. The boson-fermion correspondence gives the isomorphism of bosonic and fermionic Fock spaces [3]

$$H^{*}_{2m-1}\simeq\mathbb{C}[J_2, J_4,\ldots], \qquad (7)$$
$$<2m-1|a\mapsto <2m-1|aT|2m-1>.$$

4. Babelon–Bernard–Smirnov's Construction

In this section we review the results of [1]. Their discovery is that the fermionic description of the bosonic map ev_1 defined below (10) greatly simplifies the situation. Later it is understood that such structure is intimately related with quantum groups at a root of unity [6,8].

Let

$$\psi(z)=\sum_{n\in 2\mathbb{Z}+1}\psi_n z^{-n}, \quad \psi^*(z)=\sum_{n\in 2\mathbb{Z}+1}\psi^*_n z^{n}, \quad \nabla(z)=\sum_{n=1}^{\infty}\partial_{2n-1}z^{-2n}.$$

Define two operators Q and C by

$$Q=\int\frac{\mathrm{d}z}{2\pi i}\nabla(z)\psi(z),$$
$$C=\int\frac{\mathrm{d}z}{2\pi i}\psi(z)\frac{\mathrm{d}\psi(z)}{\mathrm{d}z}+$$
$$+\int\int_{|z_2|>|z_1|}\frac{\mathrm{d}z_1}{2\pi i}\frac{\mathrm{d}z_2}{2\pi i}\log\left(1-\left(\frac{z_1}{z_2}\right)^2\right)\nabla(z_1)\nabla(z_2)\psi(z_1)\psi(z_2).$$

Here the simple integral signifies to take the coefficient of z^{-1} and the double integral signifies to take that of $(z_1z_2)^{-1}$ when the integrand is expanded at the region $|z_1/z_2|<1$. These operators are maps of the following spaces

$$Q:\mathcal{D}\otimes H^{*}_{n}\longrightarrow\mathcal{D}\otimes H^{*}_{n+2},$$
$$C:\mathcal{D}\otimes H^{*}_{n}\longrightarrow\mathcal{D}\otimes H^{*}_{n+4}.$$

They satisfy

$$[Q,C]=0, \quad Q^2=0. \qquad (8)$$

Let us introduce new variables $\bar{S}$ by

$$\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}J_{2k}z^{-2k}\right)=\sum_{n=0}^{\infty}\bar{S}_{2n}z^{-2n}.$$

By specifying the degrees as $\deg J_{2k}=2k$, $\bar{S}_{2n}$ is a homogeneous polynomial of J_{2k}'s of degree $2k$ and has the form

$$\bar{S}_{2n}=\frac{-J_{2n}}{2n}+\cdots,$$

where $\cdots$ part does not contain J_{2n}. In particular we have the isomorphism of polynomial rings

$$\mathbb{C}[J_2, J_4, \ldots] \simeq \mathbb{C}[\bar{S}_2, \bar{S}_4, \ldots]. \tag{9}$$

The composition of isomorphism (7) and (9) gives the isomorphism

$$H^*_{-1} \simeq \mathbb{C}[\bar{S}_2, \bar{S}_4, \ldots].$$

We identify these two spaces by this isomorphism. We define a map

$$\mathrm{ev}_1 : \mathcal{D} \otimes \mathbb{C}[\bar{S}_2, \bar{S}_4, \ldots] \longrightarrow A \tag{10}$$

by

$$P(\partial) \otimes \bar{S}_2^{\alpha_2} \bar{S}_4^{\alpha_4} \cdots \mapsto P(\partial)(S_2^{\alpha_2} S_4^{\alpha_4} \cdots).$$

Then Babelon, Bernard and Smirnov have proved

THEOREM 1. [1,2]

$$Q(\mathcal{D} \otimes H^*_{-3}) + C(\mathcal{D} \otimes H^*_{-5}) \subset \operatorname{Ker} \mathrm{ev}_1.$$

They conjectured

CONJECTURE 1. *The map ev_1 is surjective.*

We set

$$B=\frac{\mathcal{D} \otimes H^*_{-1}}{Q(\mathcal{D} \otimes H^*_{-3}) + C(\mathcal{D} \otimes H^*_{-5})}.$$

Since $Q^2=0$ we have the complex

$$\cdots \xrightarrow{Q} \mathcal{D} \otimes H^*_{-5} \xrightarrow{Q} \mathcal{D} \otimes H^*_{-3} \xrightarrow{Q} \mathcal{D} \otimes H^*_{-1} \longrightarrow 0. \tag{11}$$

Since C and Q commute, it induces the following complex;

$$\cdots \xrightarrow{Q} \frac{\mathcal{D} \otimes H^*_{-5}}{C(\mathcal{D} \otimes H^*_{-9})} \xrightarrow{Q} \frac{\mathcal{D} \otimes H^*_{-3}}{C(\mathcal{D} \otimes H^*_{-7})} \xrightarrow{Q} \frac{\mathcal{D} \otimes H^*_{-1}}{C(\mathcal{D} \otimes H^*_{-5})} \longrightarrow 0. \tag{12}$$

Finally, we have the complex

$$\cdots \xrightarrow{Q} \frac{\mathcal{D}\otimes H^*_{-5}}{C(\mathcal{D}\otimes H^*_{-9})} \xrightarrow{Q} \frac{\mathcal{D}\otimes H^*_{-3}}{C(\mathcal{D}\otimes H^*_{-7})} \xrightarrow{Q} \frac{\mathcal{D}\otimes H^*_{-1}}{C(\mathcal{D}\otimes H^*_{-5})} \longrightarrow B \longrightarrow 0, \tag{13}$$

where the map to B is the natural projection.

PROPOSITION 1. (i) *For $n \geq 0$*

$$\frac{\mathcal{D}\otimes H^*_{-2n-1}}{C(\mathcal{D}\otimes H^*_{-2n-5})}$$

is a free $\mathcal{D}$-module.

(ii) *The complex* (13) *is exact.*

By this proposition (13) gives a $\mathcal{D}$-free resolution of B.

The statement (i) of this proposition is proved in [2]. We recall the change of fermions used there for further use. In the component form Q and C are written as

$$Q = \sum_{n=1}^{\infty} \partial_{2n-1}\psi_{-(2n-1)},$$

$$C = \sum_{n=1}^{\infty}\left(2(2n-1)\psi_{2n-1} - \sum_{l=1}^{\infty} P_{n,l}(\partial)\psi_{2n-1-l}\right)\psi_{-(2n-1)},$$

where we set

$$P_{n,l}(\partial) = \sum_{i+j=l+1,\, j<n} \frac{1}{n-j}\partial_{2i-1}\partial_{2j-1}.$$

We define

$$\tilde{\psi}_{-(2n-1)} = \psi_{-(2n-1)} \quad \text{for } n\geq 1,$$

$$\tilde{\psi}_{2n-1} = 2(2n-1)\psi_{2n-1} - \sum_{l=1}^{\infty} P_{n,l}(\partial)\psi_{2n-1-2l} \quad \text{for } n \geq 1,$$

Write these relations as

$$\tilde{\psi}_i = \sum_{j\in 2\mathbb{Z}+1} d_{ij}\psi_j,$$

and set $D = (d_{ij})$ which is an invertible triangular matrix. Set

$$D' = (d'_{ij}) = {}^t(D^{-1}),$$

$$\tilde{\psi}^*_i = \sum_j d'_{ij}\psi^*_j.$$

Then $\{\tilde{\psi}_i, \tilde{\psi}_j^*\}$ satisfy the canonical anti-commutation relations (6). Moreover the vacuums $<m|$, $|m>$ for $\{\psi_i, \psi_j^*\}$ become the vacuums for $\{\tilde{\psi}_i, \tilde{\psi}_j^*\}$. We denote the Fock spaces of $\{\tilde{\psi}_i, \tilde{\psi}_j^*\}$ by $\tilde{H}_m$, $\tilde{H}_m^*$. Then

$$Q = \sum_{n=1}^{\infty} \partial_{2n-1} \tilde{\psi}_{-(2n-1)}, \quad C = \sum_{n=1}^{\infty} \tilde{\psi}_{2n-1} \tilde{\psi}_{-(2n-1)},$$

and we have isomorphisms

$$\mathcal{D} \otimes H^*_{-2n-1} \simeq \mathcal{D} \otimes \tilde{H}^*_{-2n-1},$$
$$\frac{\mathcal{D} \otimes H^*_{-2n-1}}{C(\mathcal{D} \otimes H^*_{-2n-5})} \simeq \mathcal{D} \otimes \frac{\tilde{H}^*_{-2n-1}}{C\tilde{H}^*_{-2n-5}},$$

for any integer n. The statement (ii) follows from the following lemmas in a similar manner to Theorem 4.3 of [8]. The lemmas can also be proved similarly to Lemma 4.4 and 4.5 of [8].

LEMMA 1. *The complex* (11) *is exact at* $\mathcal{D} \otimes H_{-2n-1}$, $n \geq 1$.

LEMMA 2. *The map*

$$C : \tilde{H}_{-2m-1} \longrightarrow \tilde{H}_{-2m+3}$$

is injective for $m \geq 0$.

COROLLARY 1. *If we assume Conjecture* 1, *then* $A \simeq B$ *and* (13) *gives a* $\mathcal{D}$*-free resolution of* A,

$$\cdots \xrightarrow{Q} \mathcal{D} \otimes \frac{\tilde{H}^*_{-5}}{C\tilde{H}^*_{-9}} \xrightarrow{Q} \mathcal{D} \otimes \frac{\tilde{H}^*_{-3}}{C\tilde{H}^*_{-7}} \xrightarrow{Q} \mathcal{D} \otimes \frac{\tilde{H}^*_{-1}}{C\tilde{H}^*_{-5}} \xrightarrow{ev_1} A \longrightarrow 0. \tag{14}$$

Proof. Conjecture 1 implies that the map $B \to A$ induced from ev_1 is surjective. Then the injectivity follows by comparing characters. For a graded vector space $V = \oplus V_n$ with $\dim V_n < \infty$ we define the character of V by

$$\mathrm{ch}\, V = \sum q^n \dim V_n.$$

For H_n^* and $\tilde{H}_n^*$ we assign

$$\deg \psi_n = n, \quad \deg \psi_n^* = -n, \quad \deg < 2m-1| = m^2, \quad \deg \partial_i = i.$$

Then

$$\mathrm{ch}\, H^*_{-2m+1} = \mathrm{ch}\, \tilde{H}^*_{-2m+1} = \frac{q^{m^2}}{\prod_{i=1}^{\infty} (1-q^{2i})}.$$

For A we define $\deg u^{(i)} = 2+i$. Then

$$\mathrm{ch}\, A = \frac{1-q}{\prod_{i=1}^{\infty}(1-q^i)}$$

and the map ev_1 preserves grading. Using the free resolution (13) of B we have

$$\mathrm{ch}\, B = \frac{1-q}{\prod_{i=1}^{\infty}(1-q^i)} = \mathrm{ch}\, A,$$

which completes the proof. □

5. Another Construction of Free Resolution

In this section, we shall generalize the construction of [11] to the case of infinite degrees of freedom.

Let $\tau(t) = \tau(t_1, t_3, \ldots)$ be the tau function of the KdV-hierarchy [3]. We set

$$\zeta_i = \partial_i \log \tau(t), \quad \zeta_{ij} = \partial_i \partial_j \log \tau(t), \quad \partial_i = \frac{\partial}{\partial t_i}.$$

Notice that ζ_{ij} can be expressed as a differential polynomial of $u = \zeta_{11}$ and thereby is contained in A. Then

$$d\zeta_i = \sum_{j:\mathrm{odd}} \zeta_{ij} \mathrm{d}t_j$$

is a 1-form with the coefficients in A.

Let

$$\alpha_{2n-1} = \tilde{\psi}_{-(2n-1)}, \quad \beta_{2n-1} = \tilde{\psi}_{2n-1}, \quad n \geq 1.$$

Then

$$Q = \sum_{n=1}^{\infty} \partial_{2n-1} \alpha_{2n-1}, \quad C = \sum_{n=1}^{\infty} \beta_{2n-1} \alpha_{2n-1}.$$

For $N \geq 1$ set

$$\tilde{H}^*_{-1}(N) = \sum_{k=0}^{N} \sum \mathbb{C} < -2N-1 | \alpha_{i_{N-k}} \ldots \alpha_{i_1} \beta_{j_k} \ldots \beta_{j_1},$$

where the second summation is over all odd integers satisfying

$$2N+1 > i_{N-k} > \cdots > i_1 \geq 1, \quad j_k > \cdots > j_1 \geq 1.$$

Set $H^*_{-1}(0) = \mathbb{C} < -1|$. We use the notation like

$$\alpha_I = \alpha_{i_{N-k}} \ldots \alpha_{i_1},$$

for $I = (i_{N-k}, \ldots, i_1)$.

For $N < N'$ we have the inclusion

$$\begin{aligned}&\tilde{H}^*_{-1}(N) \subset \tilde{H}^*_{-1}(N'),\\ &\quad < -2N-1|a \mapsto < -2N'-1|\alpha_{2N'-1}\alpha_{2N'-3}\ldots\alpha_{2N+1}a.\end{aligned} \tag{15}$$

Thus $\{\tilde{H}^*_{-1}(N)\}$ defines an increasing filtration of H^*_{-1}:

$$\tilde{H}^*_{-1} = \cup_{N=0}^{\infty}\tilde{H}^*_{-1}(N).$$

We define a map of $\mathcal{D}$-modules

$$\mathrm{ev}_2 : \mathcal{D} \otimes \tilde{H}^*_{-1} \longrightarrow A,$$

as follows.

Let

$$\mathrm{vol} = \cdots \wedge \mathrm{d}t_5 \wedge \mathrm{d}t_3 \wedge \mathrm{d}t_1, \quad \Omega^{\frac{\infty}{2}} = \mathbb{C}\,\mathrm{vol},$$

and $\Omega^{\frac{\infty}{2}-p}$ be the vector space generated by differential forms which are obtained from vol by removing p $\mathrm{d}t_i$'s. We define the action of $\mathcal{D}$ on $A \otimes \Omega^{\frac{\infty}{2}}$ by

$$P(F\mathrm{vol}) = P(F)\mathrm{vol}, \quad P \in \mathcal{D}, \quad F \in A,$$

where we omit the tensor symbol for simplicity. Then we have the isomorphism of $\mathcal{D}$-modules

$$\begin{aligned}&A \otimes \Omega^{\frac{\infty}{2}} \simeq A,\\ &\quad F\mathrm{vol} \mapsto F.\end{aligned} \tag{16}$$

Let $v \in \tilde{H}^*_{-1}(N)$ be of the form

$$v = < -2N-1|\,\alpha_I\beta_J, \quad I = (i_{N-k}, \ldots, i_1), \quad J = (j_k, \ldots, j_1),$$

and $P \in \mathcal{D}$. We define

$$\mathrm{ev}_2(P \otimes v)\,\mathrm{vol} = P(\cdots \wedge \mathrm{d}t_{2N+3} \wedge \mathrm{d}t_{2N+1} \wedge \mathrm{d}t_I \wedge \mathrm{d}\zeta_J),$$

where $\mathrm{d}t_I = \mathrm{d}t_{i_{N-K}} \wedge \cdots \wedge \mathrm{d}t_{i_1}$ etc. We can write ev_2 more explicitly using certain determinants. Write

$$\cdots \wedge \mathrm{d}t_{2N+3} \wedge \mathrm{d}t_{2N+1} \wedge \mathrm{d}t_I \wedge d\zeta_J = F_{IJ}\,\mathrm{vol}, \quad F_{IJ} \in A.$$

Let

$$I^c = \{1, 3, \ldots, 2N-1\} \backslash I = \{l_k > \cdots > l_1\}$$

and $\mathrm{sgn}(I, I^c)$ be the sign of the permutation

$$(2N-1, \ldots, 3, 1) \longrightarrow (I, I^c).$$

Then

$$\begin{aligned} &\mathrm{ev}_2(P\otimes v)=P(F_{IJ}),\\ &\qquad F_{IJ}s=\mathrm{sgn}(I,I^c)\det(\zeta_{l_a j_b})_{1\le a,b\le k}. \end{aligned}$$

One can immediately check, using (15), that the definition of ev_2 does not depend on the choice of N such that $v\in\tilde{H}^*_{-1}(N)$.

PROPOSITION 2. (i) $\mathrm{ev}_2\left(Q(\mathcal{D}\otimes\tilde{H}^*_{-3})\right)=0.$
(ii) $\mathrm{ev}_2\left(C(\mathcal{D}\otimes\tilde{H}^*_{-5})\right)=0.$

Proof.

(i) We define

$$d\colon A\otimes\Omega^{\frac{\infty}{2}-p}\longrightarrow A\otimes\Omega^{\frac{\infty}{2}-p+1},$$

by

$$\mathrm{d}(F\otimes w)=\sum_{n=1}^{\infty}\partial_{2n-1}F\otimes w\wedge\mathrm{d}t_{2n-1},\quad F\in A,\quad w\in\Omega^{\frac{\infty}{2}-p}.$$

Then

$$\begin{aligned} &d^2=0,\quad \mathrm{d}\,\Omega^{\frac{\infty}{2}-p}=0,\\ &\mathrm{d}(w_1\wedge w_2)=w_1\wedge\mathrm{d}w_2+(-1)^q\mathrm{d}w_1\wedge w_2,\quad w_1\in A\otimes\Omega^{\frac{\infty}{2}-p-q},\quad w_2\in A\otimes\Omega^q, \end{aligned}$$

where Ω^q is the space of q-forms of $\mathrm{d}t_1$, $\mathrm{d}t_3$,... and $\mathrm{d}w_2$ is defined in an obvious manner.

For $I=(i_k,\dots,i_1)$ we set $|I|=k$. Let

$$v=<-2N-1|\alpha_I\beta_J\in\tilde{H}^*_{-3},\quad |I|+|J|=N-1,$$

and $P\in\mathcal{D}$. Then

$$Q(P\otimes v)=\sum_{n=1}^{\infty}\partial_{2n-1}P\otimes<-2N-1|\alpha_I\beta_J\alpha_{2n-1},$$

and

$$\begin{aligned} \mathrm{ev}_2\,(Q(P\otimes v))\,\mathrm{vol}&=\sum_{n=1}^{\infty}\partial_{2n-1}P(\cdots\wedge\mathrm{d}t_{2N+1}\wedge\mathrm{d}t_I\wedge d\zeta_J\wedge\mathrm{d}t_{2n-1})=\\ &=P\left(\mathrm{d}(\cdots\wedge\mathrm{d}t_{2N+1}\wedge\mathrm{d}t_I\wedge\mathrm{d}\zeta_J)\right)=\\ &=P\Big((-1)^{N-1}\mathrm{d}(\cdots\wedge\mathrm{d}t_{2N+1})\wedge\mathrm{d}t_I\wedge d\zeta_J+\\ &\quad+\cdots\wedge\mathrm{d}t_{2N+1}\wedge\mathrm{d}(\mathrm{d}t_I\wedge\mathrm{d}\zeta_J)\Big)=\\ &=0. \end{aligned}$$

(ii) Let

$$v = P \otimes < -2N-1|\alpha_I \beta_J \in \mathcal{D} \otimes \tilde{H}^*_{-5}.$$

Then

$$\begin{aligned}
\mathrm{ev}_2(Cv)\mathrm{vol} &= \mathrm{ev}_2\left(-P \otimes \sum_{n=1}^{\infty} < -2N-1|\alpha_I \beta_J \alpha_{2n-1}\beta_{2n-1}\right) = \\
&= -P\left(\cdots \wedge \mathrm{d}t_{2N+1} \wedge \mathrm{d}t_I \wedge \mathrm{d}\zeta_J \wedge \sum_{n=1}^{\infty} \mathrm{d}t_{2n-1} \wedge \mathrm{d}\zeta_{2n-1}\right) = \\
&= 0,
\end{aligned}$$

since

$$\sum_{n=1}^{\infty} \mathrm{d}t_{2n-1} \wedge \mathrm{d}\zeta_{2n-1} = \sum_{n,m=1}^{\infty} (\partial_{2m-1}\partial_{2n-1} \log \tau) \mathrm{d}t_{2n-1} \wedge \mathrm{d}t_{2m-1} = 0.$$

□

By (ii) of Proposition 2 we have a map of $\mathcal{D}$-modules

$$\mathcal{D} \otimes \frac{\tilde{H}^*_{-1}}{C\tilde{H}^*_{-5}} \xrightarrow{ev_2} A. \tag{17}$$

Notice that the proof of Proposition 2 is much simpler than that of Theorem 1 in [1,2].

The following theorem is proved in the next section.

THEOREM 2. *If we assume Conjecture 1 then the map (17) is surjective. In particular if we replace* ev_1 *by* ev_2 *in (14) then it gives a* $\mathcal{D}$*-free resolution of* A.

6. Relation of Two Constructions

In this section, we show

THEOREM 3. *The map* ev_1 *is surjective if and only if the map* ev_2 *is surjective.*

The remaining part of this section is devoted to the proof of this theorem. Let us set

$$\begin{aligned}
\omega_{n,m} &= \mathrm{ev}_1(\bar{\omega}_{n,m}), \quad \bar{\omega}_{n,m} = < -1|\psi_n \psi^*_{-m} T| - 1 >, \quad m, n \geq 1, \\
\omega_n &= \sum_{m:\mathrm{odd}} \omega_{n,m} dt_m.
\end{aligned}$$

LEMMA 3. *The* 1-*form* ω_n *is closed.*

Proof. By Theorem 1

$$\mathrm{ev}_1\left(<-1|\psi_n\psi^*_{-m_1}\psi^*_{-m_2}QT|-1>\right)=0. \tag{18}$$

We substitute

$$Q=\sum_{i:\text{odd}}\partial_i\psi_{-i}$$

into Equation (18) and get

$$\partial_{m_2}\omega_{n,m_1}-\partial_{m_1}\omega_{n,m_2}=0$$

which proves the lemma. □

By the lemma ω_n should be written as $d\eta_n$ for some function η_n which is not necessarily in A. We shall find the explicit form of η_n and study its properties.

Let

$$\Psi(z)=\frac{\tau(t-[z^{-1}])}{\tau(t)}\mathrm{e}^{\xi(t,z)},\quad \xi(t,z)=\sum_{n=1}^{\infty}t_{2n-1}z^{2n-1},$$

be the wave function of the KdV-hierarchy [2,3], where $[z^{-1}]=\left(z^{-1},\frac{z^{-3}}{3},\frac{z^{-5}}{5},\ldots\right)$. Set

$$S(z)=\sum_{n=0}^{\infty}S_{2n}z^{-2n},\quad S_0=1.$$

Notice that $S_{2n}=\partial_1\partial_{2n-1}\log\tau(t)$. Then, by the bilinear identity for $\tau(t)$ [3], we have (for example, see [2, Remark 1, p. 391])

$$S(z)=\frac{\tau(t-[z^{-1}])\tau(t+[z^{-1}])}{\tau(t)^2}. \tag{19}$$

Define $X(z)$ by [2, p. 388]

$$X(z)=\frac{-1}{2}\log S(z)+\log\Psi(z).$$

Using Equation (19) we have

$$X(z)=\frac{1}{2}\left(\log\tau(t-[z^{-1}])-\log\tau(t+[z^{-1}])\right)+\xi(t,z). \tag{20}$$

Let us set

$$\eta(z)=z^{-1}\left(-X(z)+\xi(t,z)\right).$$

By Equation (20) $\eta(z)$ is expanded into negative even powers of z,

$$\eta(z)=\sum_{n=1}^{\infty}\eta_{2n-1}z^{-2n}.$$

Then

PROPOSITION 3. (i) $\mathrm{d}\eta_{2n-1}=\omega_{2n-1}$.
(ii) *We have*

$$\eta_{2n-1}=\frac{1}{2n-1}\zeta_{2n-1}+a_{2n-1}, \tag{21}$$

for some $a_{2n-1}\in A$.

Proof. We use the following lemma [2, p. 392 Remark 3].

LEMMA 4. *The following equation holds.*

$$\nabla(w)X(z)=\frac{z}{w^2-z^2}\frac{S(w)}{S(z)},\quad |w|>|z|.$$

By this lemma we get

$$\nabla(w)\eta(z)=\frac{1}{z^2-w^2}\left(\frac{S(w)}{S(z)}-1\right),\quad |w|>|z|. \tag{22}$$

Since the right hand side of this equation is regular at $z^2=w^2$, Equation (22) is valid at $|z|>|w|$. Let $\bar{S}(z)=\sum_{n=0}^{\infty}\bar{S}_{2n}z^{-2n}$ with $\bar{S}_0=1$. Using

$$T\psi(z)T^{-1}=\bar{S}(z)\psi(z),\quad T\psi^*(z)T^{-1}=\bar{S}(z)^{-1}\psi^*(z),$$
$$<-1|\psi(z)\psi^*(w)|-1>=\frac{zw}{z^2-w^2},\quad |z|>|w|,$$

we have

$$<-1|\psi(z)\psi^*(w)T|-1>=\frac{\bar{S}(w)}{\bar{S}(z)}\frac{zw}{z^2-w^2},\quad |z|>|w|.$$

On the other hand, expanding in z and w, we have

$$<-1|\psi(z)\psi^*(w)T|-1>=\sum_{m,n=1}^{\infty}\bar{\omega}_{2n-1,2m-1}z^{-2n+1}w^{-2m+1}+\frac{zw}{z^2-w^2},$$
$$|z|>|w|.$$

Thus

$$\frac{1}{z^2-w^2}\left(\frac{\bar{S}(w)}{\bar{S}(z)}-1\right)=(zw)^{-1}<-1|\psi(z)\psi^*(w)T|-1>-\frac{1}{z^2-w^2}$$
$$=\sum_{m,n=1}^{\infty}\bar{\omega}_{2n-1,2m-1}z^{-2n}w^{-2m}.$$

Taking ev_1 we have

$$\partial_{2m-1}\eta_{2n-1}=\omega_{2n-1,2m-1}.$$

□

Proof of Theorem 3. We denote by $\mathrm{vol}(i_1,\ldots,i_k)$ the differential form which is obtained from vol by removing $\mathrm{d}t_{i_1},\ldots,\mathrm{d}t_{i_k}$.

Suppose that ev_1 is surjective. Notice that

$$\mathrm{ev}_1:\mathcal{D}\otimes H^*_{-1}\longrightarrow A,$$

is given by

$$\mathrm{ev}_1\left(P\otimes<-1|\psi_{2i_1-1}\ldots\psi_{2i_k-1}\psi^*_{2j_k-1}\ldots\psi^*_{2j_1-1}\right)=P\left(\det(\omega_{2i_a-1,2j_b-1})_{1\le a,b\le k}\right).$$

Thus the space

$$\left(\frac{A}{\sum_{n=1}^{\infty}\partial_{2n-1}A}\right)\mathrm{vol} \tag{23}$$

is generated by all elements of the form

$$\mathrm{vol}(i_1,\ldots,i_k)\wedge\omega_{2j_1-1}\wedge\cdots\wedge\omega_{2j_k-1}, \tag{24}$$

as a vector space over $\mathbb{C}$. We substitute Equation (21) into (24). Then the space (23) is generated by the elements of the form

$$\mathrm{vol}(i_1,\ldots,i_k)\wedge d\zeta_{2j_1-1}\wedge\cdots\wedge d\zeta_{2j_k-1},$$

since the terms containing da_{2r-1} belong to the denominator $\left(\sum_{n=1}^{\infty}\partial_{2n-1}A\right)\mathrm{vol}$. Since (23) generates A over $\mathcal{D}$, ev_2 is surjective. The converse is similarly proved.

□

As a corollary of Theorem 3, Theorem 2 in the previous section is proved.

7. Concluding Remarks

From the view point of conformal field theories and their integrable deformations [5,14,15] the space A corresponds to the descendent fields of the identity operator. It is possible to consider the spaces corresponding to descendents of other primary fields [1]. It is an interesting problem to construct their $\mathcal{D}$-free resolutions. In the quantum case free resolutions are constructed for those spaces [1,8]. In particular the spaces become free modules in "odd cases". The classical and even the finite dimensional cases are expected to have similar structures. Geometrically, to consider non-identity primary fields corresponds to consider the spaces of sections of certain non-trivial flat line bundles over affine Jacobians in stead of affine rings [11].

References

1. Babelon, O., Bernard, D., Smirnov, F.: Null-vectors in integrable field theory. Commun. Math. Phys. **186**, 601–648 (1997)
2. Babelon, O., Bernard, D., Talon, M.: Introduction to Classical Integrable Systems. Cambridge University Press, London (2003)
3. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for Soliton equations, Nonlinear Integrable Systems-Classical Theory and Quantum Theory. In: Jimbo, M., Miwa, T. (eds.) World Scientific, Singapore 1983
4. Donagi, R., Markman, E.: Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles. Lect. Notes Math. **1620**, 1–119 (1996)
5. Eguchi, T., Yang, S.K.: Deformations of conformal field theories and soliton equations. Phys. Lett. B **224**, 373–378 (1989)
6. Feigin, B., Jimbo, M., Kashiwara, M., Miwa, T., Mukhin, E., Takeyama, Y.: A functional model for the tensor product of level 1 highest and level -1 lowest modules for the quantum affine algebra $U_q(\widehat{sl_2})$. Eur. J. Combin. **25**, 1197–1229 (2004)
7. Nakayashiki, A.: On the cohomologies of theta divisors of hyperelliptic Jacobians. Contemp. Math. **309**, 177–183 (2002)
8. Nakayashiki, A.: The space of local fields as a module over the ring of local integrals of motion. Int. Math. Res. Not. **52**, 2797–2822 (2004)
9. Nakayashiki, A.: On hyperelliptic abelian functions of genus 3. arXiv:0809.3303
10. Nakayashiki, A.: Sigma function as a tau function. arXiv:0904.0846
11. Nakayashiki, A., Smirnov, F.: Cohomologies of affine hyperelliptic Jacobi varieties and integrable systems. Commun. Math. Phys. **217**, 623–652 (2001)
12. Nakayashiki, A., Smirnov, F.: Euler characteristics of theta divisors of Jacobians for spectral curves. In: Kuznetsov, V.B. (ed.) CRM Proceedings and Lecture Notes, vol. 32, pp. 239–246 (2002)
13. Smirnov, F., Zeitlin, V.: Affine Jacobians of Spectral Curves and Integrable Models. arXiv:math-ph/020303

14. Sasaki, R., Yamanaka, I.: Virasoro algebra, vertex operators, quantum sine-Gordon and solvable quantum field theories. Adv. Stud. Pure Math. **16**, 271–296 (1988)
15. Zamolodchikov, A.B.: Integrable field theory from conformal field theory. Adv. Stud. Pure Math. **19**, 641–674 (1989)